WEST BENGAL STATE UNIVERSITY

B.Sc. Honours PART-I Examinations, 2018

MATHEMATICS-HONOURS

PAPER-MTMA-I

Time Allotted: 4 Hours — Full Marks: 100

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Group-A

Answer any *five* questions from the following — $5\times 5 = 25$

1. (a) Let a and b be two positive integers such that G.C.D of $a, b = (a,\, b) = 1$. Prove that $(ac,\, b) = (c,\, b)$. — 2

   (b) Let $a > 1$ and $m,\, n$ are positive integers. Prove that $(a^m - 1,\, a^n - 1) = a^{(m,n)} - 1$. — 3

2. If p is prime then prove that there exist no positive integers a and b such that $a^2 = pb^2$. — 5

3. Solve: $111x \equiv 75 \pmod{321}$. — 5

4. (a) Expand $\cos 5\theta$ in powers of $\cos\theta$. — 2

   (b) Find all the values of $(-i)^{\frac{3}{4}}$. — 3

5. Express i^i in the form $a+ib$, with $a,\, b$ reals. Use it to find the value of $\sin(\log i^i)$ in integer. — 2+3

6. If $z = x+iy$ then prove that $|\sinh y| \le \max(|\sin z|, |\cos z|) \le \cosh y$. — 5

7. (a) Find $f(x-3)$ if $f(x-2) = 4x^4 + 3x^2 - x + 2$. — 2

   (b) If the expression $x^3 + 3px^2 + 6qx + r$ and $x^2 + 2px + 2q$ have a common factor, show that $4(p^2 - 2q)(4q^2 - pr) = (r - 2pq)^2$. — 3

8. If a, b, c, are roots of $x^3 + qx + r = 0$, form the equation whose roots are $\dfrac{b^2+c^2}{a^2}$, $\dfrac{a^2+c^2}{b^2}$, $\dfrac{b^2+a^2}{c^2}$. — 5

9. Use Strum's function to show that the roots of the equation $x^4 + 5x^3 - 13x + 5 = 0$ are all real and distinct. — 5

Group-B

Answer any *two* questions from the following 10×2 = 20

10. (a) Let A, B, C be three non-empty subsets of a set S. Then prove that, $(A \setminus B)\times C = (A\times C)\setminus(B\times C)$. 3

(b) A relation ρ on the set of real numbers $\mathbb{R}$ is defined as follows: $a\,\rho\, b$ if and only if $|a| \le b$. Show that ρ is transitive but neither reflexive nor symmetric. 3

(c) Define injective and surjective mappings. 1+3

Let $f: A\to B$, $g: B\to C$, $h: B\to C$ be three mappings such that f is surjective and $g\circ f = h\circ f$. Prove that $g = h$.

11. (a) Let $\mathbb{N}$ be the set of all positive integers. Let R be the relation on $\mathbb{N}$ defined by $R = \{(a, b)\in \mathbb{N}\times\mathbb{N} : a - b \le 0\}$. 3

Prove that R is a partial order relation on $\mathbb{N}$.

(b) If $f: S\to T$ is one-one onto, then prove that $f^{-1}: T\to S$ is one-one onto. 3

(c) Prove that a semigroup is a group if it a quasigroup. 4

12. (a) In a group G, prove that $(a^{-1})^{-1} = a$, where $a\in G$ and hence show that group of even order contains an element of order 2. 3

(b) In an Abelian group G, prove that $(ab)^n = a^n b^n$ for all $a, b\in G$, where n is an integer. 3

(c) Prove that anon-empty subset H of a group G is a subgroup of G if and only if for all $a, b\in H$, $a^{-1}b\in H$. 4

13. (a) If R is ring with unity 1, then show that R has characteristic n if and only if $n\cdot 1 = 0$. 3

(b) Prove that every finite integral domain is a field. 3

(c) Show that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b\in\mathbb{Q}\}$ is a subfield of the field $\mathbb{R}$ of real numbers, where $\mathbb{Q}$ is the set of rational numbers. 4

Group-C

Answer any *three* questions from the following 5×3 = 15

14. If A be a skew-symmetric matrix and $(I + A)$ be a non-singular matrix, then show that $B = (I - A)(I + A)^{-1}$ is orthogonal, where I is the identity matrix of the same size as A. 5

15. Compute the inverse of the matrix A by using row operations, where, 5

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 4 & 2 & -1 \\ 7 & 3 & 1 \end{bmatrix}.$$

16. Find the non-singular matrices P and Q such that PAQ is in the normal form and hence find the rank of the matrix 4+1

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}.$$

17. Apply Laplace's method along second and third rows to prove that 5

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2 .$$

18. For a square matrix A of order n, prove that $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)I_n$, where I_n denotes the identity matrix of order n. 5

19. Reduce the quadratic form $2x^2 - 4yz + 6zx + y^2 + z^2$ into its normal form and obtain the rank, signature and nature of the form. 5

Group-D

Answer any *one* question from the following 10×1 = 10

20. (a) A company has a manufacturing unit producing two types of products, product A and B. At a time only one type of production is possible. It requires 35 minutes to produce one unit of product A and 25 minutes to produce one unit of product B. The raw material required is 1.4 kgs for one unit of product A and 2.0 kgs for one unit of product B. The factory can run 35 hours per week and the raw material available is 140 kg per week. The profit for the products are Rs. 180 and Rs. 230 per unit of product A and product B respectively. Formulate an LPP to maximize the profit. 5

(b) Solve graphically the LPP: 5

Maximize: $z = 2x_1 + 3x_2$,

Subject to $4x_1 + 3x_2 \le 12$

$4x_1 + x_2 \le 8$

$x_1 - x_2 \ge -3,\ \ x_1, x_2 \ge 0$.

21. (a) Verify whether $x_1 = 1, x_2 = 2, x_3 = 1$ is a feasible solution of the system of equations: 1+4

$$2x_1 + 3x_2 + 5x_3 = 13$$

$$3x_1 - x_2 + 3x_3 = 4$$

Reduce it to a basic feasible solution, if possible.

(b) Show that (2, 0, 1) is a solution of the system of equations 1+2+2

$$2x_1 + 3x_2 + 4x_3 = 8$$

$$x_1 + 3x_2 + 2x_3 = 4$$

Justify whether this is a basic solution. Find another solution of the system which is a basic solution, state basic and non-basic variables.

Group-E

Section-I

Answer any *three* questions from the following 5×3 = 15

22. Reduce the equation $5x^2 + 4xy + 8y^2 - 14x - 20y - 19 = 0$ to its canonical form and hence identify the conic represented by it. 5

23. Show that the product of the perpendicular distances from the point (x_1, y_1) to the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$. 5

24. Show that the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is right angled if $(a+b)(al^2 + 2hlm + bm^2) = 0$. 5

25. If the straight lines $r\cos(\theta - \alpha_i) = p_i$, $i = 1, 2, 3$ are concurrent, prove that $p_1\sin(\alpha_2 - \alpha_3) + p_2\sin(\alpha_3 - \alpha_1) + p_3\sin(\alpha_1 - \alpha_2) = 0$. 5

26. Show that the equation of a circle passing through the pole can be written in the form $r = A\cos\theta + B\sin\theta$, where A, B are constants. 5

Section-II

Answer any *three* questions from the following 5×3 = 15

27. Show that the straight lines whose direction cosines are given by $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$. 5

28. Find the mirror image of the point (2, –1, 3) about the plane $3x - 2y + 7z = 0$. 5

29. Find the equation of the plane passing through the point (2, 5, –8) and perpendicular to each of the planes $2x - 3y + 4z + 1 = 0$ and $4x + y - 2z + 6 = 0$. 5

30. A variable plane with constant distance p from the origin cuts the coordinate axes at A, B, C. Three planes are drawn through the points A, B, C parallel to the coordinate planes. Show that the locus of the points of intersection is given by $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} = \dfrac{1}{p^2}$. 5

31. Find the equation of the straight line passing through the origin and interest both the lines 5

$$3x + 2y + 4z - 5 = 0 = 2x - 3y + 4z + 1 \text{ and}$$
$$2x - 4y + z + 6 = 0 = 3x - 4y + z - 3.$$

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Group-A

Answer any ***five*** questions from the following 5×5 = 25

1. State and prove Cantor's theorem on nested intervals. 1+4

2. (a) Let A and B be two non-empty bounded sets of real numbers and $C=\{x+y : x\in A,\, y\in B\}$. Show that $\sup C = \sup A + \sup B$. 2+3

   (b) Show that every bounded sequence has a convergent subsequence.

3. (a) For any subset $A\subset R$, prove that $(A')'\subset A'$ where A' denotes the set of all limit points of A. 3+2

   (b) For any two subsets $A, B\subset R$, show that the equality $(A\cap B)' = A'\cap B'$ does not hold in general.

4. (a) State Cauchy's second limit theorem. Using it find the limit 3+2

$$\lim_{n\to\infty}\frac{\{(n+1)(n+2)\ldots(2n)\}^{\frac{1}{n}}}{n}.$$

   (b) Evaluate : $\lim\limits_{x\to\infty}\dfrac{[x]}{x}$, if exists.

5. (a) Show that the sequence $\{x_n\}$ converges to 1 where $x_n=\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}$. 2+3

   (b) Let A be a nonempty subset of R and $d(x, A)=\inf\{|x-y| : y\in A\}$. Prove that $d(x, A)=0$ if and only if $x\in\overline{A}$.

6. (a) Prove that a convergent sequence of real numbers is a Cauchy sequence. 2+3

(b) Show that the sequence $\{x_n\}$ is not convergent where $x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, $n \geq 1$.

7. (a) Prove that union of two denumerable sets is denumerable. 2+3

(b) Prove that no nonempty proper subset of R is both open and closed in R.

8. (a) Let $D \subset R$ and f, g, h, be three function defined on D to R. Let $c \in D'$. If $f(x) \leq g(x) \leq h(x)$ for all $x \in D-\{c\}$ and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = l$ then prove that $\lim_{x\to c} g(x) = l$. 3+2

(b) Show that $\lim_{x\to 0} \sqrt{x}\, \sin\frac{1}{x} = 0$.

9. Let $t: R \to R$ be a continuous function and $f(x+y) = f(x)+f(y)$ for all $x, y \in R$. If $f(1) = k$ prove that $f(x) = kx$ for all $x \in R$. 5

Group-B

10. Answer any ***two*** questions from the following: 4×2 = 8

(a) Evaluate $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$. 4

(b) If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, prove that $I_n = \frac{1}{n-1} - I_{n-2}$ 2+2

Hence find the value of I_5.

(c) (i) Prove that $\frac{B(m, n+1)}{n} = \frac{B(m,n)}{m+n}$ 2+2

(ii) Evaluate $\int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$.

11. Answer any ***three*** questions from the following: 4×3 = 12

(a) Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where the parameters a, b are connected by the relation $a+b=c$, c being a nonzero constant.

(b) Find all the asymptotes of the curve $x^2y + xy^2 + xy + y^2 + 3x = 0$.

(c) Show that the curve $(x+y)^3 - \sqrt{2}\,(y-x+2) = 0$ has a double point at (–1, 1). Find the equation of the tangents at that point and identify the nature of the double point.

(d) If ρ_1 and ρ_2 are the radii of curvature at the ends of conjugate diameter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then prove that $\rho_1^{2/3}+\rho_2^{2/3}=\frac{a^2+b^2}{(ab)^{2/3}}$.

(e) Determine the pedal equation of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to a focus. Where $a>b$.

Group-C

Answer any ***three*** questions from the following 10×3 = 30

12.(a) Examine whether the equation $(\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy = 0$ is exact or not and then solve it. 5

(b) Find the orthogonal trajectories of the cardiodes $r = a(1-\cos\theta)$. 5

13.(a) Reduce the equation $x^2(y - px) = p^2 y$ to Clairaut's form by putting $x^2 = u$ and $y^2 = v$. Hence obtain the general and singular solution. 5

(b) Solve the following differential equation $y = (1+p)x + ap^2$. 5

14.(a) Solve by the method of undetermined coefficient: $(D^2+4)y = x^2\sin 2x$. 5

(b) Solve : $(x^2D^2 - 3xD + 5)\,y = x^2\sin(\log x)$. 5

15.(a) Solve by the method of variant of parameters: 5

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}.$$

(b) Solve by reducing to a linear equation: $(1+x^2)\frac{dy}{dx} - 4x^2\cos^2 y + x\sin 2y = 0$. 5

16.(a) Solve: $x^4\frac{d^3y}{dx^3} + 3x^3\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 2xy = \log x$. 5

(b) Solve: $\left(\frac{d^2y}{dx^2} + y\right)\cot x + 2\left(\frac{dy}{dx} + y\tan x\right) = \sec x$ by reducing it to normal form. 5

17.(a) Solve by the method of operational factors: 5

$$x\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} - y = x^2.$$

(b) Solve: $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$ by changing the independent variable. 5

Group-D

Answer any ***five*** questions from the following $5\times5 = 25$

18. If $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $\vec{a}\times\vec{b}\times\vec{c} = \frac{1}{2}\vec{b}$, find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$; $\vec{b}$, $\vec{c}$ being non parallel.

19. Show by vector method, that the straight line joining the mid points of two non-parallel sides of a trapezium are parallel to the parallel sides and half of their sum in length.

20. For any three vectors $\vec{a}, \vec{b}, \vec{c}$, prove that $[\vec{a}+\vec{b}\;\; \vec{b}+\vec{c}\;\;\; \vec{c}+\vec{a}] = 2[\vec{a}\;\; \vec{b}\;\; \vec{c}]$.

21.(a) Forces $\vec{P}$, $\vec{Q}$ act at O and have a resultant $\vec{R}$. If any transversal cuts lines of action of $\vec{P}$, $\vec{Q}$ and $\vec{R}$ at A, B, C respectively, then show that $\frac{|\vec{P}|}{OA}+\frac{|\vec{Q}|}{OB}=\frac{|\vec{R}|}{OC}$. 3

(b) A particle acted on by constant forces $4\hat{i}+5\hat{j}-3\hat{k}$ and $3\hat{i}+2\hat{j}+4\hat{k}$ is displaced from the point $\hat{i}+3\hat{j}+\hat{k}$ to the point $2\hat{i}-\hat{j}-3\hat{k}$. Find the total work done by the forces. 2

22.(a) Find the moment of the force $4\hat{i}+2\hat{j}+\hat{k}$ acting at a point $5\hat{i}+2\hat{j}+4\hat{k}$ about the point $3\hat{i}-\hat{j}+3\hat{k}$. 3

(b) Find the vector equation of the plane passing through the origin and parallel to the vectors $2\hat{i}+3\hat{j}+4\hat{k}$ and $4\hat{i}-5\hat{j}+4\hat{k}$. 2

23.(a) Find the constants a, b, c so that $\vec{V} = (x+2y+az)\,\hat{i}+(bx-3y-z)\,\hat{j}+(4x+cy+2z)\,\hat{k}$ is irrotational. 3

(b) Find a so that $\vec{V} = 3x\hat{i}+(x+y)\hat{j}-ax\,\hat{k}$ is solenoidal. 2

24. Show that the necessary and sufficient condition that a non-zero vector $\vec{u}$ always remains parallel to a fixed line is that $\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}$.

25.(a) If $f(r)$ is differentiable, then prove that $\nabla^2 f(r) = f''(r)+\frac{2}{r}\,f'(r)$. 3

(b) Show that $\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{\vec{a}}{r^3}+\frac{3(\vec{a}\cdot\vec{r})\vec{r}}{r^5}$, where $\vec{a}$ is a constant vector. 2

26. $\vec{V} = 2xz^2\hat{i}-yz\hat{j}+3xz^3\hat{k}$ and $\phi = x^2yz$, then find

(a) curl $(\phi\vec{V})$ 3

(b) curl curl$\vec{V}$. 2

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