WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2020

MTMACOR08T-MATHEMATICS (CC8)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5 = 10$

(a) Let $f(x) = c, \quad 0 \le x \le c$

$\quad = 2c, \quad c < x \le 1.$

If $\int_0^1 f(x)\,dx = \frac{7}{16}$, find the value of c.

(b) Let $f:[0,1] \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{n}, \quad \frac{1}{n+1} < x \le \frac{1}{n}, \quad n \in N,$$
$$= 0, \quad x = 0.$$

Show that f is Riemann integrable.

(c) Show that the integral $\int_1^\infty \frac{\sin x}{\sqrt{x+x^3}}\,dx$ is absolutely convergent.

(d) Assuming convergence of the integral, evaluate $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$.

(e) For $n \in \mathbb{N}$, $f_n(x) = x^n$, $x \in [0,1)$. Find the limit function of $\{f_n\}$ and check the validity of $\lim\limits_{x\to1}\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\lim\limits_{x\to1} f_n(x)$.

(f) For $n \in \mathbb{N}$, $f_n(x) = \frac{x^n}{1+x^n}$, $x \in \left[0, \frac{3}{2}\right]$. Find the limit function of $\{f_n\}$ and check the continuity of the limit function. Is the convergence uniform?

(g) Show that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges uniformly on $\mathbb{R}$.

(h) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{2.5.8...(3n-1)} x^n$.

2. (a) If $f:[a,b] \to \mathbb{R}$ be a bounded function. Prove that f is Riemann integrable over $[a, b]$ if and only if for any $\varepsilon > 0$ there is a partition P of $[a, b]$ such that 4

$$U(P, f) - L(P, f) < \varepsilon.$$

(b) Give an example with proper justification of a Riemann integrable function which has no primitive. 4

3. (a) Examine the convergence of $\int_0^1 x^{p-1}\log x\,dx$ for $p>0$. 4

(b) Apply Dirichlet's test to show that $\int_0^\infty \cos(x^2)\,dx$ is convergent. 4

4. (a) If $D\subset\mathbb{R}$ and each function $f_n : D\to\mathbb{R}$ of the sequence of functions $\{f_n\}$ be continuous on D and $\{f_n\}$ converges uniformly to f on D then prove that f is continuous on D. 3

(b) Show that the sequence of functions f_n defined on [0, 1] by 2+2+1

$$\begin{aligned} f_n(x) &= x(1-nx)\,, \quad 0\le x<\frac{1}{n} \\ &= \quad 0\;, \qquad \frac{1}{n}\le x\le 1 \end{aligned}$$

converges to the function f given by $f(x)=0,\ x\in[0,\ 1]$. Show that $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \ne \int_0^1 f(x)\,dx$. Is the convergence of the sequence uniform?

5. (a) Let the power series $\sum_{n=0}^{\infty} a_n x^n$ converge at a point $c\ne 0$. Show that the series converges absolutely for all $x\in\mathbb{R}$ such that $|x|<|c|$. 4

(b) Assuming $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$ for $-1<x<1$, obtain the power series expansion for $\tan^{-1}x$. Also deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$. 3+1

6. Show that the function defined by 4+2+2

$$f(x)=(\pi-|x|)^2, \qquad x\in[-\pi,\pi]$$

satisfies the Dirichlet's condition in $[-\pi,\pi]$. Obtain the Fourier series of $f(x)$ in $[-\pi,\pi]$. Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

7. (a) Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if $m>0,\ n>0$. 5

(b) Show that $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\ d\theta = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$. 3

8. (a) A function f is defined on [0, 1] by 2+2

$$f(x) = (-1)^{n-1} \quad \text{when } \frac{1}{n+1} < x \le \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$
$$= 0 \quad \text{when } x = 0.$$

Prove that f is integrable on [0, 1] and $\int_0^1 f = \log \frac{4}{e}$.

(b) Show that $\frac{\pi^3}{96} < \int_{-\pi/2}^{\pi/2} \frac{x^2}{5+3\sin x}\,dx < \frac{\pi^3}{24}$. 4

9. (a) The sequence of continuous functions $\{h_n\}$ is uniformly convergent on $[a, b]$ and $g_n(x) = \int_a^x h_n(x)\,dx,\ a \le x \le b$. Prove that the sequence $\{g_n\}$ is uniformly convergent on $[a, b]$. 4

(b) Examine the uniform convergence of the sequence of functions $\{g_n\}$ where for each $n \in \mathbb{N}$, g_n is defined by $g_n(x) = \frac{nx}{1+n^3x^2}$, $x \in [0, 1]$. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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