WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 2nd Semester Examination, 2021

MTMACOR03T-MATHEMATICS (CC3)

REAL ANALYSIS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Show that for every real number x, there is a positive integer n such that $n > x$.

(b) Find the derived set of the set $S=\{(-1)^m+\frac{1}{n}\ ;\ m, n \in \mathbb{N}\}$, where $\mathbb{N}$ denotes the set of natural numbers.

(c) For a set S in $\mathbb{R}$, show that the interior S° of S is the largest open contained in $\mathbb{R}$.

(d) For each $x \in (0, 2)$, let $I_x = \left(\frac{x}{2}, \frac{x+2}{2}\right)$. Show that the family $\mathcal{G} = \{I_x : x \in (0, 2)\}$ is an open cover of the set $S = \{x \in \mathbb{R} : 0 < x < 2\}$. Show that no finite subfamily of $\mathcal{G}$ can cover S.

(e) Prove that a convergent sequence is bounded.

(f) Give an example of an unbounded set having exactly one limit point.

(g) If a sequence $(a_n)_n$ of positive real numbers converges to zero, then prove that the sequence has a subsequence $(a_{n_k})_k$ such that the series $\sum_{k=1}^{\infty} a_{n_k}$ is convergent.

(h) Use comparison test (limit form) to test the convergence of the series

$$\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\cdots\cdots$$

(i) Examine whether the following series converges:

$$1-\frac{2^2}{2\,!}+\frac{3^3}{3\,!}-\frac{4^4}{4\,!}+\cdots\cdots$$

2. (a) Find $\sup A^c$ and $\inf A^c$, where $A=\{x \in \mathbb{R} : x^2 - x - 12 \geq 0\}$. 3

(b) Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then prove that the set $T = \{-x \ :\ x \in S\}$ is bounded below and $\inf T = -\sup S$. 3

(c) If A and B be two nonempty bounded subsets of $\mathbb{R}$ and if $C=\{x+y\ :\ x\in A,\ \ y\in B\}$, then show that $\sup C=\sup A+\sup B$. 2

3. (a) Examine whether the set $\{x\in\mathbb{R}:\ \sin x=0\}$ is countable. 2

(b) Show that an infinite set has a countably infinite subset. 3

(c) What is meant by neighbourhood of a point in $\mathbb{R}$? Check whether the set $\left\{\frac{1}{n}:n\in\mathbb{N}\right\}\cup\{0\}$ is a neighbourhood of 0 or not. 1+2

4. (a) If A be a nonempty bounded subset of $\mathbb{R}$, show that $A'\cap(b,\infty)=\phi$, where $b=\sup A$ and A' is the derived set of A. 2

(b) Prove that every bounded infinite subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$. 3

(c) Show that the set $S=\{x\in\mathbb{R}:|x-1|+|x-2|<3\}$ is an open set. 3

5. (a) Give an example of a bounded set in $\mathbb{R}$ which is not compact and a closed set in $\mathbb{R}$, which is not compact (with justifications). 2+2

(b) Define closed set in $\mathbb{R}$, in terms of its limit points. Hence, show that 1+1+2

(i) intersection of arbitrary collection of closed sets in $\mathbb{R}$ is also closed in $\mathbb{R}$, and

(ii) there is no proper nonempty subset of $\mathbb{R}$ which is both open and closed.

6. (a) State sequential definition of compact sets in $\mathbb{R}$. Hence, show that a set in $\mathbb{R}$ is compact if and only if it is closed and bounded. 1+3

(b) Let E be a closed and bounded subset of $\mathbb{R}$. Prove that every open cover of E has a finite subcover. 4

7. (a) Define convergence of a sequence in $\mathbb{R}$. 1

(b) Prove that a sequence in $\mathbb{R}$ may converge to at most one limit in $\mathbb{R}$. 2

(c) Let $(x_n)_n$ be a sequence of real numbers and let $x\in\mathbb{R}$. If $(a_n)_n$ is a sequence of positive real numbers with $\lim a_n=0$ and if for some constant $c>0$, there exists $m\in\mathbb{N}$ such that $|x_n-x|\le c\,a_n$, for all $n\ \ge\ m$ then, from the definition of convergence of sequence, prove that $\lim x_n=x$. 2

(d) Using the result stated in (c) above, prove that $\lim\,(x^{1/n})=1$ for any real number $x>0$. 3

8. (a) If a sequence $(a_n)_n$ converges to zero and also if the sequence $(b_n)_n$ is bounded, then show that the sequence $(a_nb_n)_n$ converges to zero. 3

(b) Let $A(\subseteq\mathbb{R})$ be dense in $\mathbb{R}$. Then, for every $a\in\mathbb{R}$, prove that there is a sequence $(a_n)_n$ of elements in A, which converges to a. 2

(c) Prove that the sequence $\{x_n\}$ where $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$, $\forall n \geq 1$, is convergent and converges to $\sqrt{2}$. 3

9. (a) Let $(x_n)_n$ be a sequence of real numbers defined by $x_1 = \frac{1}{3}$, $x_{2n} = \frac{1}{3}x_{2n-1}$ and $x_{2n+1} = \frac{1}{3} + x_{2n}$ for $n = 1, 2, 3, \cdots\cdots$. Find $\liminf x_n$ and $\limsup x_n$. 3

(b) Prove that every Cauchy sequence in $\mathbb{R}$ is convergent. 2

(c) Show that the ratio test is not suitable for arriving at any conclusion about convergence of the series 3

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\cdots$$

Examine the convergence of this series by applying root test.

10.(a) Use Cauchy's integral test to show that the series $\sum \frac{1}{n(\log n)^p}$, $p > 0$, converges for $p > 1$ and diverges for $p \leq 1$. 4

(b) Let $a \in \mathbb{R}$ with $a > 0$. Show that the series 4

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots\cdots \text{ is}$$

(i) absolutely convergent if $p > 1$, and

(ii) conditionally convergent if $0 < p \leq 1$.

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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