WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 3rd Semester Examination, 2021-22

MTMACOR06T-MATHEMATICS (CC6)

Time Allotted: 2 Hours Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Let $(S, *)$ be a semigroup which has exactly one idempotent element. If for each $a \in S$, there exists $b \in S$ such that $a * b * a = a$, then prove that $(S, *)$ is a group.

(b) Let G be a commutative group. Prove that $H = \{a \in G : o(a) \text{ divides } 10\}$ is a subgroup of G. [$o(a)$ denotes the order of the element $a \in G$.]

(c) Suppose that G is a finite cyclic group and 8 divides $|G|$ (order of the group G). How many elements of order 8 does G have? Justify your answer. If a is an element of order 8, list the other elements of G of order 8.

(d) Let G be a group and H be a subgroup of G. Let $a \in G - H$. Then prove that $aH \cap H = \phi$.

(e) Determine whether the permutation $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$ is odd or even.

(f) Let G be a group and H be a subgroup of G such that $aba^{-1}b^{-1} \in H$ for all $a, b \in G$. Prove that H is a normal subgroup of G.

(g) Prove that every Abelian group of order 39 is cyclic.

(h) Let $\mathbb{C}^*$ denote the multiplicative group of all non-zero complex numbers. Show that the mapping $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ defined by $f(x) = x^6$, for all $x \in \mathbb{C}^*$, is a group-homomorphism. Determine kernel of f.

(i) Let H and K be subgroups of S_6 generated by $\sigma = (1\ 2)\ (3\ 4\ 5)$ and $\tau = (1\ \ 3\ 2\ \ 4\ \ 6\ \ 5)$ respectively. Are these subgroups isomorphic to each other? Justify your answer.

2. (a) Examine whether the set $S = \left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : x, y \in \mathbb{R} \setminus \{0\} \text{ and } x + y \neq 0 \right\}$ forms a group with respect to matrix multiplication. 2

(b) Let $(G, \circ)$ be a semigroup containing a finite number of elements in which both the cancellation laws hold. Show that $(G, \circ)$ is a group. 4

(c) Let G be a group. If $a^{-1}b^2(bab)^{-1}ba^2 = b$ for all $a, b \in G$, prove that the group G is Abelian. 2

3. (a) Let G be a group with the identity element e and $a \in G$ be such that $o(a) = n$. If $a^m = e$ for some positive integer m, prove that n is a divisor of m. 3

(b) Let $(G, \cdot)$ be a group and $a, b \in G$ be two elements of finite order. If $a \cdot b \cdot a^{-1} = b$ and $\gcd(o(a), o(b)) = 1$, then show that $o(a \cdot b) = o(a) \cdot o(b)$. 4

(c) Determine the order of the element [36] in the additive group $\mathbb{Z}_{56}$ of integers modulo 56. 1

4. (a) Let G be a group and A be a non-empty subset of G. Define the centralizer $C_G(A)$ of A in G. Prove that $C_G(A)$ is a subgroup of G. 1+3

(b) Let H and K be two subgroups of a group $(G, \circ)$. If $K \circ H$ is a subgroup of G, then prove that $H \circ K = K \circ H$. 2

(c) Let H be a subgroup of a finite group G. Suppose that g is an element of G and n is the smallest positive integer such that $g^n \in H$. Prove that n divides the order of g. 2

5. (a) Let G be a cyclic group of order n. If m is a positive integer such that m divides n, then prove that G has a unique subgroup of order m. 3

(b) Let G be a group of order 63. Show that G has a non-trivial subgroup. 2

(c) Find all cyclic subgroups of the group $(\mathbb{Z}_5, +)$. 3

6. (a) Show that the number of even permutations in the symmetric group S_n of degree $n \geq 2$ is the same as that of odd permutations. 3

(b) Let $\beta = (1 \quad 2 \quad 3)\ (1 \quad 4 \quad 5)$ and $\alpha = \beta^{99}$ in S_5. Write α in cycle notation and hence examine whether α is an even permutation. 3

(c) Let $\sigma = (1 \quad 2)\ (4 \quad 5)\ (6 \quad 7)$ and $\gamma = (2 \quad 5 \quad 6)\ (1 \quad 3 \quad 4 \quad 7)$ be two permutations in S_7. Compute $\sigma^{-1}\gamma\sigma$. 2

7. (a) Let H and K be two finite subgroups of a group G. Prove that $|HK| = \dfrac{|H||K|}{|H \cap K|}$. 5

(b) Let H and K be two subgroups of a group G. If $|H| = 63$ and $|K| = 45$, then by using Lagrange's theorem for finite groups show that $H \cap K$ is Abelian. 3

8. (a) Show that the external direct product $\mathbb{Z}_6 \times \mathbb{Z}_4$ of the cyclic groups $\mathbb{Z}_6$ and $\mathbb{Z}_4$ is not a cyclic group. 3

(b) Define a normal subgroup of a group. 1

(c) Prove that a subgroup H of a group G is a normal subgroup of G if and only if $gHg^{-1} \subseteq H$, for all $g \in G$. 4

9. (a) State Cauchy's theorem for finite Abelian groups. 1

(b) Let $Z(G)$ denote the center of a group G. If $G/Z(G)$ is cyclic, then show that the group G is Abelian. 4

(c) Let G be a group such that $|G| = 165$ and $|Z(G)| = 15$. Find the number of elements of G of order 3 and the number of elements of order 55. 3

10.(a) (i) Let G be a group. For each $a \in G$, define a mapping $\tau_a : G \to G$ by $\tau_a(g) = ag$ for all $g \in G$. Show that $\tau_a \in A(G)$, where $A(G)$ denotes the group of all permutations on the set G. 2

(ii) Show that the mapping $\psi : G \to A(G)$ defined by $\psi(a) = \tau_a$, for all $a \in G$, is a homomorphism from the group G to the permutation group $A(G)$. Hence, by using first isomorphism theorem, prove that G is isomorphic to a subgroup of the permutation group $A(G)$. 2+2

(b) Prove that two infinite cyclic groups are isomorphic. 2

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within* 1 *hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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