WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2022

MTMACOR10T-MATHEMATICS (CC10)

RING THEORY AND LINEAR ALGEBRA-I

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Show that the ring $\left\{\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbb{Z}\right\}$ does not contain unity.

(b) Solve $x^3 = x$ in the ring $\{\mathbb{Z}_6, +, .\}$ considering the equation over that ring.

(c) In the ring $R = \{f / f : [0,1] \to \mathbb{R}\}$ w.r.t. usual addition and multiplication of functions, show that, for any fixed point $c \in [0,1]$ the set $I_c = \{f \in R / f(c) = 0\}$ forms an ideal.

(d) Let $f : R \to S$ be a homomorphism from a ring R to a ring S. Show that $f(-a) = -f(a)\ \forall\, a \in R$.

(e) State First Isomorphism Theorem for Rings.

(f) Write down a basis of the vectorspace $\mathbb{R}^3$ over $\mathbb{R}$, containing $(2, 3, 4)$ as a basis vector.

(g) Examine if $\{(x, y) \in \mathbb{R}^2 : x^2 + y = 0\}$ is a subspace of the vectorspace $\mathbb{R}^2$ over $\mathbb{R}$.

(h) Examine if $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x - y)\ \forall\, (x, y) \in \mathbb{R}^2$ is a linear transformation from the vectorspace $\mathbb{R}^2$ over $\mathbb{R}$ to itself.

2. (a) Find the units and the nonzero divisors of zero in the ring $\{\mathbb{Z}_{12}, +, .\}$ 2+2

(b) Examine if the ring $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$ is a field. 4

3. (a) Show that the ring $C[0,1] = \{f / f : [0,1] \to \mathbb{R}$ continuum$\}$ is a ring with unity. Is $C[0,1]$ an integral domain? Justify. 2+2

(b) Show that the intersection of two ideals of a ring is an ideal of that ring but union of two ideals of a ring may not be an ideal of that ring. 2+2

4. (a) Suppose that $\{R,+,.\}$ is a ring with the property $a\cdot a=a \;\; \forall\, a\in R$. Show that R is commutative and every element in R is self-inverse w.r.t. '+'. 2+2

(b) Show that the field $\mathbb{Q}$ has no proper subfield. 2

(c) Find all units of $\mathbb{Z}\,[i]$. 2

5. Determine all possible ring homomorphisms from 2+2+2+2

(a) $\mathbb{Z}\to\mathbb{Z}$

(b) $\mathbb{Z}_3\to\mathbb{Z}_6$

(c) $\mathbb{Z}_6\to\mathbb{Z}_3$

(d) $\mathbb{Z}\to\mathbb{Z}_6$

6. (a) In the ring $\mathbb{Z}_{24}$, show that $I=\{[0],[8],[16]\}$ is an ideal. Find all elements of the quotient ring $\mathbb{Z}_{24}/I$. 4

(b) Define linearly independent set in a vectorspace V over $\mathbb{R}$ and show that any nonempty subset of a linearly independent set in a vectorspace V over $\mathbb{R}$ is again linearly independent. 2+2

7. (a) Show that $S=\{(x,y,z)\in\mathbb{R}^3/\,x+2y+z=0$ and $2x+y+3z=0\}$ is a subspace of the vectorspace $\mathbb{R}^3$ over $\mathbb{R}$ and find a basis of S. 2+2

(b) Determine all possible subspaces of the vectorspaces $\mathbb{R}^3$ over $\mathbb{R}$ and $\mathbb{R}^2$ over $\mathbb{R}$. 2+2

8. Let V and W be vectorspaces over $\mathbb{R}$ and $T:V\to W$ be a linear transformation.

(a) Define kernel of T. 2

(b) Show that $\ker T$ is singleton set iff T is injective and in this case, image of any linearly independent subset of V is a linearly independent subset of W. 2+2+2

9. (a) Show that a linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ is injective iff it is surjective. 2+2

(b) Show that the function $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by $T(x,y,z)=(x-y,\,x+2y,\,y+3z)$ $\forall\,(x,y,z)\in\mathbb{R}^3$ is an invertible linear transformation and verify whether $T^{-1}(x,y,z)=\left(\frac{2x+y}{3},\frac{y-x}{3},\frac{x-y+3z}{9}\right)$ $\forall\,(x,y,z)\in\mathbb{R}^3$. 2+2

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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