WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 2nd Semester Examination, 2021

MTMACOR04T-MATHEMATICS (CC4)

ORDINARY DIFFERENTIAL EQUATION AND VECTOR CALCULUS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* questions from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Show that if $f(x, y)$ satisfies the condition $\left|\frac{\partial f}{\partial y}\right| \le \lambda$, for all (x, y) in a given domain D, then for the same constant λ the Lipschitz's condition is also satisfied in D.

(b) State Picard's theorem for the existence and uniqueness of the solution of a first order and first degree ordinary differential equation.

(c) Show that if y_1 and y_2 are two linearly independent solutions of the linear differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

then the Wronskian $W(y_1, y_2)$ is given by $W(y_1, y_2) = A \cdot e^{-\int P\,dx}$, A is a non-zero constant.

(d) Give an example to show that the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

may not be unique although $f(x, y)$ is continuous.

(e) Locate and classify the singular points of the equation

$$(3x+1)\,x\frac{d^2y}{dx^2} - (x+1)\frac{dy}{dx} + 3y = 0$$

(f) Find the value of d when the vectors $(2\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} + 2\hat{j} + d\hat{k})$ and $(3\hat{i} - 4\hat{j} + 5\hat{k})$ are coplanar.

(g) If $\vec{A}(t) = t^2\hat{i} + (t-1)\hat{j} - 4\hat{k}$, then find $\int_1^2 \vec{A}(t)\,dt$.

(h) If $\vec{r}=(\sin t)\hat{i}+(\cos t)\hat{j}+2t\,\hat{k}$, then show that $\left|\frac{d}{dt}\vec{r}\right|=\sqrt{5}$ and $\left|\frac{d^2}{dt^2}\vec{r}\right|=1$.

(i) If $\vec{a}=3t\,\hat{i}+4t\hat{j}-t^3\hat{k}$, $\vec{b}=t^2\hat{i}+3t\,\hat{k}$, find $\frac{d}{dt}(\vec{a}\cdot\vec{b})$.

2. (a) Show that $\frac{d}{d\theta}\{\vec{a}\times(\vec{b}\times\vec{c})\}=\hat{i}+3\hat{j}+9\hat{k}$ at $\theta=\frac{\pi}{2}$ 5

where $\vec{a}=(\sin\theta)\hat{i}+(\cos\theta)\hat{j}+3\hat{k}$,

$\vec{b}=(\cos\theta)\hat{i}-(\sin\theta)\hat{j}-3\hat{k}$

$\vec{c}=2\hat{i}+3\hat{j}-\hat{k}$

(b) If $\vec{r}=(a\cos t)\,\hat{i}+(a\sin t)\,\hat{j}+(at\tan\alpha)\,\hat{k}$, then prove that 3

$$\left[\frac{d\vec{r}}{dt}\;\frac{d^2\vec{r}}{dt^2}\;\frac{d^3\vec{r}}{dt^3}\right]=a^3\tan\alpha$$

3. (a) If $\vec{r}=t\,\hat{i}-t^2\hat{j}+(t-1)\,\hat{k}$; $\vec{s}=2t^2\,\hat{i}+6t\,\hat{k}$, find $\int_0^2\vec{r}\cdot\vec{s}\,dt$ and $\int_0^2\vec{r}\times\vec{s}\,dt$. 4

(b) Show that the four points $A(\vec{a}), B(\vec{b}), C(\vec{c}), D(\vec{d})$ are coplanar if and only if $[\vec{b}\;\vec{c}\;\vec{d}]+[\vec{c}\;\vec{a}\;\vec{d}]+[\vec{a}\;\vec{b}\;\vec{d}]=[\vec{a}\;\vec{b}\;\vec{c}]$. 4

4. (a) Prove that $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ if and only if $(\vec{c}\times\vec{a})\times\vec{b}=\vec{0}$. 4

(b) At any instant *t*, the position vector of a moving particle in a plane, relative to some origin *O* in the plane is given by $\vec{r}=\cos(\omega t)\,\hat{i}+\sin(\omega t)\,\hat{j}$, where ω is a constant. Show that velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$, acceleration $\vec{a}$ is directed towards *O* and has magnitude proportional to the distance of the particle from *O* and $\vec{r}\times\vec{v}$ is a constant vector. What is the physical interpretation of the motion? 4

5. (a) Solve the simultaneous linear differential equation 5

$$\frac{dx}{dt}+4x+3y=\sin t,\quad \frac{dy}{dt}+2x+5y=e^t$$

by operator method.

(b) Find the general solution for the system 3

$$\frac{dy_1}{dt}=-y_1+5y_2\;,\quad \frac{dy_2}{dt}=-4y_1-5y_2$$

6. Solve $2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$ in powers of $(x-1)$. 8

7. (a) Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=10\sin x$, where at $x=0,\ \ y=0$ and $\frac{dy}{dx}=2$. 4

(b) Solve by the method of variation of parameters: $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=e^x\sin x$ 4

8. (a) Solve $(2+x)^2\frac{d^2y}{dx^2}-4(2+x)\frac{dy}{dx}+6y=x$. 4

(b) Prove that two solutions $y_1(x)$ and $y_2(x)$ of the equation 4

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0,\quad a_0(x)\neq 0,\ \ x\in(a,b)$$

are linearly dependent if and only if their Wronskian is identically zero.

9. (a) If $\frac{d^2x}{dt^2}+2h\frac{dx}{dt}+(h^2+p^2)\,x=ke^{-ht}\cos pt$, then prove that 5

$$x=c_1e^{-ht}\cos(pt+c_2)+\frac{k}{2p}te^{-ht}\sin pt$$

where $c_1,\ c_2$ are arbitrary constants.

(b) Solve $\frac{d^2r}{dt^2}-\omega^2r=-g\sin\omega t$. 3

10.(a) If $\vec{r}(t)=5t^2\,\hat{i}+t\,\hat{j}-t^3\hat{k}$, then prove that 4

$$\int_1^2\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=-14\hat{i}+75\hat{j}-15\hat{k}$$

(b) Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\frac{e^{-x}}{x^2}$ by the methods of variation of parameters. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within* 1 *hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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