WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2021

MTMACOR08T-MATHEMATICS (CC8)

Time Allotted: 2 Hours — Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Prove that $f:[0,3]\to\mathbb{R}$ defined by $f(x)=x+[x]$ is integrable.

(b) Give an example, with proper justifications, of a discontinuous function which has a primitive.

(c) Show that the integral $\int_0^1 \frac{1}{\sqrt{x}}\sin\frac{1}{x}\,dx$ is absolutely convergent.

(d) Evaluate $\int_0^{\pi/2} \sin^{3/2}\theta\cos^3\theta\,d\theta$, assuming convergence of the given integral.

(e) Examine whether the sequence of functions $\{f_n\}$ converges uniformly on $\mathbb{R}$, where for all $n\in\mathbb{N}$,

$$f_n(x)=\frac{x+nx^2}{n},\quad x\in\mathbb{R}$$

(f) Find the limit function $f(x)$ of the sequence $\{f_n\}$ on [0, 1], where for all $n\in\mathbb{N}$,

$$f_n(x)=\begin{cases} nx \;; & 0\le x\le \frac{1}{n} \\ 1 \;; & \frac{1}{n}< x\le 1 \end{cases}$$

Hence, state with reason whether $\{f_n\}$ converges uniformly on [0, 1].

(g) Show that the series $\sum_{n=1}^{\infty}\frac{1}{n^3+n^2x^2}$ is uniformly convergent on $\mathbb{R}$.

(h) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty}(2+(-1)^n)^n x^n$.

(i) Show that the series $\sum_{n=1}^{\infty}\frac{x\sin(n^2x)}{n^2}$ converges to a continuous function on [0, 1].

2. (a) If a function $f:[a,b]\longrightarrow\mathbb{R}$ be integrable and $f(x)\geq 0$ for $x\in[a,b]$ and there exists a point $c\in[a,b]$, such that f is continuous at c with $f(c)>0$, then prove that $\int_a^b f>0$. 4

(b) Let f be continuous on $[a, b]$ and for each α, β, $a\leq\alpha<\beta\leq b$, 4

$$\int_\alpha^\beta f(x)\,dx=0$$

Prove that f is identically zero on $[a, b]$.

3. (a) If a function $f:[a,b]\longrightarrow\mathbb{R}$ be bounded and for every $c\in(a,b)$, f is integrable on $[c, b]$, then prove that f is integrable on $[a, b]$. 4

(b) Give an example of a function $f:[0,1]\longrightarrow\mathbb{R}$ which is integrable on $[c, 1]$, $0<c<1$ but not integrable on $[0, 1]$. 4

4. (a) Let $f_n: D\rightarrow\mathbb{R}$ be bounded functions on $D\subseteq\mathbb{R}$, for all $n\in\mathbb{N}$ so that the sequence of functions $\{f_n\}$ is uniformly convergent to $f: D\rightarrow\mathbb{R}$. Show that f is bounded on D. 3

(b) Find the limit function $f(x)$ of the sequence $\{f_n\}$, where for all $n\in\mathbb{N}$, 5

$$f_n(x)=\frac{nx}{1+nx},\quad x\in[0,1]$$

Justify that $\{f_n\}$ is not uniformly convergent on $[0, 1]$. Further show that $f(x)$ is Riemann integrable on $[0, 1]$ and

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\int_0^1 f(x)\,dx$$

5. (a) Let $\{a_n\}$ be a convergent sequence of real numbers and let $\{f_n\}$ be a sequence of functions satisfying 3

$$\sup\{|f_n(x)-f_m(x)|: x\in A\}\leq|a_n-a_m|,\ n,\ m\in\mathbb{N}$$

Prove that $\{f_n\}$ converges uniformly on A.

(b) If 5

$$f_n(x)=\frac{1}{2n^2}\log(1+n^4x^2)\ ,\quad x\in[0,1],\ n\in\mathbb{N},$$

then prove that $\{f_n'(x)\}$ converges pointwise but not uniformly to $f'(x)$ on $[0, 1]$, where f is the uniform limit function of $\{f_n\}$.

6. (a) Let $f_n:D\longrightarrow\mathbb{R}$ be a continuous function on D, for $n\in\mathbb{N}$. If the series $\sum_{n=1}^{\infty}f_n$ be uniformly convergent on D, then prove that the sum function S is continuous on D. 4

(b) Study the continuity on $[0,\infty)$ of the function f defined by 4

$$f(x)=\sum_{n=1}^{\infty}\frac{x}{((n-1)x+1)(nx+1)}$$

7. (a) Let the series $\sum_{n=1}^{\infty} f_n(x)$, $x \in A$, converges uniformly on A and that $f: A \longrightarrow \mathbb{R}$ be bounded. Prove that the series $\sum_{n=1}^{\infty} f(x)\, f_n(x)$ converges uniformly on A. 3

(b) Let the series $\sum_{n=1}^{\infty} f_n(x)$ of continuous functions on $[a, b]$ converge uniformly on $[a, b]$ and $g(x)$ be bounded and integrable on $[a, b]$. Prove that 5

$$\int_{\alpha}^{\beta} f(x) g(x)\, dx = \sum_{n=1}^{\infty} \int_{\alpha}^{\beta} f_n(x) g(x)\, dx,$$

where $a \le \alpha < \beta \le b$, and the convergence of the series of integrals is uniform on $[a, b]$.

8. (a) Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$, where $0 < R < \infty$. Prove that the series converges uniformly on $[-r, r]$ for any $0 < r < R$. 4

(b) Let the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ be r. Find the radius of convergence of $\sum_{n=0}^{\infty} a_n x^{2n}$. 4

9. (a) Let $f(x) = \begin{cases} \dfrac{\pi}{2} + x \;, & -\pi \le x \le 0 \\ \dfrac{\pi}{2} - x \;, & 0 \le x \le \pi. \end{cases}$ 5

Show that $f(x) = \dfrac{4}{\pi} \sum_{n=1}^{\infty} \dfrac{\cos(2n-1)x}{(2n-1)^2}$, where $x \in [-\pi, \pi]$.

(b) Examine whether the series $\sum_{n=1}^{\infty} \dfrac{\sin(nx)}{\sqrt{n}}$ is a Fourier Series. 3

10. (a) Examine the convergence of the integrals $\int_2^{\infty} \dfrac{x^2}{\sqrt{x^7+1}}\, dx$ and $\int_2^{\infty} \dfrac{x^3}{\sqrt{x^7+1}}\, dx$. 4

(b) Show the convergence of $\int_0^{\infty} \left(\dfrac{x}{x+1} \right) \sin(x^2)\, dx$. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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