WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 6th Semester Examination, 2021

MTMACOR13T-MATHEMATICS (CC13)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* questions from the rest

1. Answer any ***five*** questions from the following: $2\times 5 = 10$

(a) Let (X, d) be a metric space. Prove that

$$|d(x,z)-d(z,y)|\leq d(x,y) \text{ for all } x, y, z\in X$$

(b) Let (X, d) be a discrete metric space. Prove that $\{x\}$ is an open subset of X for all $x\in X$.

(c) Let (X, d) be a metric space. Let $x, y\in X$, $x\neq y$. Prove that there exists open balls B_1 and B_2 in X so that $x\in B_1$, $y\in B_2$ and $B_1\cap B_2=\emptyset$.

(d) Let (X, d) be the metric space, where $X=(0,1)$ with d the induced metric from the standard metric on $\mathbb{R}$. Give an example, with proper justification, of a Cauchy sequence in X which is not convergent in X.

(e) Prove that $\mathbb{Q}$ is a disconnected subset of $\mathbb{R}$ with the standard metric, where $\mathbb{Q}$ denotes the set of all rational numbers.

(f) Do the Cauchy-Riemann equations hold for $f(z)=|z|^2$?

(g) Find the locus of the point z satisfying $|z-1|\geq 3$.

(h) If the sum and product of two complex numbers are both real, then show that the numbers must be both real or conjugate to each other.

(i) For which value(s) of $z=u+iv$, the function

$$\omega=e^{-v}(\cos u+i\sin u)$$

becomes non analytic?

2. (a) Let X be a non-empty set. Define a function $d: X\times X\to\mathbb{R}$ satisfying the following conditions: 3+5

(i) $d(x,y)=0$ iff $x=y$ in X

(ii) $d(x,y)\leq d(x,z)+d(y,z)$ for all $x, y, z\in X$.

Prove that d is a metric on X.

(b) Define a complete metric space. Show that $l_p (1 \le p \le \infty)$ is a complete metric space.

3. (a) State and prove Cantor's intersection theorem. 4+4

(b) For any two subsets A and B in a metric space prove that

$$\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B.$$

4. (a) Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Prove that (2+2)+4

(i) A is open in Y iff $\exists$ an open set G in X such that $A = G \cap Y$.

(ii) A is closed in Y iff $\exists$ a closed set F in X such that $A = F \cap Y$.

(b) For any non-empty subset A of a metric space (X, d) prove that $x \in \overline{A}$ iff there exists a sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$.

5. (a) Prove that $A \subset \mathbb{R}$ is connected with respect to usual metric iff it is an interval. 4

(b) Prove that a metric space (X, d) is compact iff every collection of closed subsets of X having finite intersection property has non-empty intersection. 4

6. (a) Let (X, d) and (Y, ρ) be two metric spaces and $f : X \to Y$ be a function. Prove that f is continuous iff $f^{-1}(G)$ is open in X whenever G is open in Y. 4+4

(b) Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a uniformly continuous function. Prove that if $\{x_n\}$ be a Cauchy sequence in X then $\{f(x_n)\}$ is also Cauchy sequence in Y.

7. (a) State and prove Cauchy's integral formula for the first derivative of an analytic function. 5+3

(b) Evaluate $\displaystyle\int_{|z|=1} \frac{f(z)}{z+2} dz$. Hence deduce the value of $\displaystyle\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta$.

8. (a) Let $f(z) = u(x, y) + i\, v(x, y)$ be a function defined in a region D such that u, v and their first order partial derivatives are continuous in D and first order partial derivatives of u, v satisfy Cauchy-Riemann equations at a point $(x, y) \in D$ then prove that f is differentiable at $z = x + iy$. 5+3

(b) Prove that $f(z) = \begin{cases} \dfrac{z \operatorname{Re} z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is continuous at $z = 0$ but not differentiable at $z = 0$.

9. (a) If $P(z)$ be a polynomial of degree $n \geq 1$ with real or complex coefficients then show that $P(z) = 0$ has at least one root in the complex plane. 4+4

(b) Find Taylor series expansion of $f(z) = \dfrac{z^2 - 1}{(z+3)(z+2)}$ in $|z| < 2$.

10.(a) If $f(z)$ be analytic in the interior of a circle C with centre at α and radius r, then at each point z in the interior of C, show that $f(z) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(\alpha)}{n!}(x - \alpha)^n$ 4

(b) Let $f(z)$ be an entire function so that $|f(z)| \leq M$ for all z, where M is a positive constant. Show that $f(z)$ is a constant function. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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