WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 6th Semester Examination, 2021

MTMACOR14T-MATHEMATICS (CC14)

RING THEORY AND LINEAR ALGEBRA II

Time Allotted: 2 Hours   Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Either prove or disprove : If F is a field, then $F[x]$ is a field.

(b) Show that in an integral domain D, any two gcd's of two elements, if they exists are associates.

(c) Find all associates of $1+i$ in $\mathbb{Z}[i]$.

(d) If α, β be any two vectors in a Euclidean space V, then prove that

$$\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$$

(e) Let V be an inner product space over a field $F(\mathbb{R}$ or $\mathbb{C})$ and $y, z \in V$. If $\langle x, y\rangle = \langle x, z\rangle,\ \forall x \in V$, then show that $y = z$.

(f) In Euclidean space $\mathbb{R}^3$ with standard inner product, let P be the subspace generated by the vectors (1, 1, 0) and (0, 1, 1). Find $P^{\perp}$.

(g) Find the minimum polynomial of the matrix

$$\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

(h) Let T be the linear operator on $\mathbb{R}^2$ defined by $T(a, b) = (2a+5b, 6a+b)$ and β be the standard ordered basis for $\mathbb{R}^2$. Find the characteristic polynomial of T.

(i) Let V be a finite dimensional inner product space and let T and U be linear operators on V. Show that $(TU)^* = U^*T^*$.

2. (a) Define a polynomial ring. 2+3

If D is an integral domain, show that $D[x]$ is an integral domain.

(b) Let $f(x)=x^4+[3]x^3+[2]x^2+[2]$, $g(x)=x^2+[2]x+[1]\in\mathbb{Z}_5[x]$. 3

Find $q(x), r(x)\in \mathbb{Z}_5[x]$ such that $f(x)=q(x)g(x)+r(x)$, where either $r(x)=0$ or, $0\le \deg r(x)<\deg g(x)$.

3. (a) Show that in an integral domain D, every prime element is irreducible. 3

(b) Consider the integral domain $\mathbb{Z}[i\sqrt{5}]=\{a+bi\sqrt{5}:a,b\in\mathbb{Z}\}$. Show that $3=3+0.i\sqrt{5}\in\mathbb{Z}[i\sqrt{5}]$ is irreducible but not a prime. 3

(c) Test for the irreducibility of the polynomial $x^3+[3]x+[4]$ over $\mathbb{Z}_5$. 2

4. (a) Let R be a commutative ring with 1. If $R[x]$ is a principal ideal domain, show that R is a field. 4

(b) Show that $\mathbb{Z}[x]$ is not a principal ideal domain. 1

(c) Show that in a unique factorization domain, every irreducible element is prime. 3

5. (a) Prove that the eigenvalues of a real symmetric matrix are all real. 3

(b) Find the eigenvalues and the corresponding eigen vectors of the following real matrix: 5

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

6. (a) Prove that every square matrix satisfies its own characteristic equation. 4

(b) Diagonalize the following matrix orthogonally: 4

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

7. (a) Show that an orthogonal set of non-null vectors in an Euclidean space V is linearly independent. 3

(b) Apply the Gram-Schmidt process to the vectors $\beta_1=(1,0,1)$, $\beta_2=(1,0,-1)$, $\beta_3=(0,3,4)$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product. 5

8. (a) In an inner product space V prove that $|\langle\alpha,\beta\rangle|\le\|\alpha\|\,\|\beta\|$, for all $\alpha,\beta\in V$. 4

(b) Let V be a finite dimensional inner product space and f be a linear functional on V. Then show that there exists a unique vector β in V such that $f(\alpha)=\langle\alpha,\beta\rangle$, for all $\alpha\in V$. 4

9. (a) Let V and W be vector spaces over the same field F of dimension n and m respectively. Prove that the space $L(V, W)$ has dimension mn. 4

(b) The matrix of $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by 2+2

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ relative to the standard ordered basis of $\mathbb{R}^2$.

Find T and T^*, where T^* is the adjoint of T.

10.(a) Let T be a linear operator on a vector space V. Define a T-invariant subspace of V. 2+1+1

Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(a, b, c) = (a+b, b+c, 0).$$

Show that the xy-plane $= \{(x, y, 0) : x, y \in \mathbb{R}\}$ and the x-axis $= \{(x, 0, 0) : x \in \mathbb{R}\}$ are T-invariant subspaces of $\mathbb{R}^3$.

(b) Find the dual basis of the basis $\beta = \{(2, 1), (3, 1)\}$ of $\mathbb{R}^2$. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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