WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 3rd Semester Examination, 2021-22

MTMACOR05T-MATHEMATICS (CC5)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Determine $f(1)$ so that the function $f(x)=\dfrac{x^2-1}{x-1}$, $x\neq 1$ is continuous at $x=1$.

(b) If $f(x)=\begin{cases}1+x & , \; x\leq 2\\ 5-x & , \; x>2\end{cases}$, then examine the existence of $f'(2)$.

(c) Show that $f(x)=x-[x]$ has a jump discontinuity at $x=1$.

(d) Show that $\frac{\sin x}{x}$ decreases steadily in $0<x<\frac{\pi}{2}$.

(e) Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be defined by

$$f(x)=\begin{cases}x\sin\frac{1}{x} & , \; x\neq 0\\ 0 & , \; x=0\end{cases}$$

Show that f is continuous at $x=0$.

(f) Correct or justify:

Every bounded sequence is convergent.

(g) Check whether Rolle's theorem is applicable to the function $f(x)=|x|$, $x\in[-1,1]$.

(h) Let $f(x)=(x-p)^n(x-q)^m$ in $[p, q]$. Show that there exists a $\xi\in(p,q)$ which divides $[p, q]$ in the ratio $n:m$.

(i) Show that for the function $f(x)=|x-1|$, $f'(1)$ does not exist.

2. (a) Let $f:[a,b]\longrightarrow\mathbb{R}$ be continuous in $[a, b]$. If $k\in\mathbb{R}$ satisfies $f(a)<k<f(b)$, then prove that there exists a point c between a and b such that $f(c)=k$. 5

(b) Let $f:[a,b]\to[a,b]$ be a continuous function. Show that there exists at least one $c\in[a,b]$ such that $f(c)=c$. 3

3. (a) Let $A \subseteq \mathbb{R}$, $f: A \longrightarrow \mathbb{R}$ and $c \in A$. Let f be continuous at c and let $\{x_n\}$ be a sequence in A such that $\lim_{n\to\infty} x_n = c$. Then show that $\lim_{n\to\infty} f(x_n) = f(c)$. 4

(b) A function is defined on $\mathbb{R}$ by 4

$$\begin{aligned} f(x) &= 1 \quad , \quad x \in \mathbb{Q} \\ &= 0 \quad , \quad x \in \mathbb{R} - \mathbb{Q} \end{aligned}$$

where $\mathbb{Q}$ is set of rational numbers. Prove that f is continuous at no point $c \in \mathbb{R}$.

4. (a) Give an example with proper justifications to show that a bounded function on a closed and bounded interval need not be continuous. 3

(b) Show that the function 5

$$f(x) = x^2 \quad \forall\, x \in \mathbb{R}$$

is not uniformly continuous on $\mathbb{R}$ but its restriction to any non empty bounded interval J of $\mathbb{R}$ is uniformly continuous.

5. (a) Let $f: I \to J$ be a bijective function where I, J are intervals in $\mathbb{R}$. Let f be differentiable at $d \in I$ and let $f'(d) \neq 0$. Show that f^{-1} is differentiable at $f(d)$ and $(f^{-1})'[f(d)] = [f'(d)]^{-1}$. 4

(b) Let $f: I \longrightarrow \mathbb{R}$ be a function differentiable at $c \in I$, where I is an interval in $\mathbb{R}$. Let $f'(c) > 0$. Show that there is a $\delta > 0$ so that 4

$$x \in (c-\delta, c) \cap I \;\Rightarrow f(x) < f(c)$$
$$x \in (c, c+\delta) \cap I \;\Rightarrow f(x) > f(c)$$

6. (a) Show that between any two distinct real roots of $e^x \sin x - 1 = 0$ there is at least one real root of $e^x \cos x + 1 = 0$. 3

(b) State and prove Lagrange's mean value theorem. 1+4

7. (a) Let $A \subseteq \mathbb{R}$ and $f, g, h:\ A \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ be a limit point of A. 4

If $f(x) \le g(x) \le h(x)\ \ \forall\, x \in A$, $x \neq c$ and if $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ then show that $\lim_{x\to c} g(x) = L$.

(b) Let $g: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by 4

$$g(x) = \begin{cases} x + 2x^2 \sin\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Show that g is not monotonic in any neighbourhood of zero.

8. (a) Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be defined by 4

$$f(x)=\begin{cases} x^2\sin\frac{1}{x^2} & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$$

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$.

(b) Let I be an integral and a function $f:I\longrightarrow\mathbb{R}$ be differentiable at $c\in I$. Then show that f is continuous at c. Is the converse true? Justify your answer. 4

9. (a) Show that $\frac{\tan x}{x}>\frac{x}{\sin x}$, $0<x<\frac{\pi}{2}$. 5

(b) Let $f:[0,1]\longrightarrow\mathbb{R}$ be a continuous function which is differentiable on (0, 1). Show that the equation 3

$$f(1)-f(0)=\frac{f'(x)}{3x^2}$$

has at least one solution in (0, 1).

10.(a) Write with proper justification, Maclaurin's infinite series expansion for 4

$$f(x)=\sin x \ , \quad x\in\mathbb{R}$$

(b) Find the maximum and minimum values of $y=\sin x(1+\cos x)$, $0\leq x\leq 2\pi$. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within* 1 *hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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