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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 4th Semester Examination, 2023

MTMHGEC04T/MTMGCOR04T-MATHEMATICS (GE4/DSC4)

ALGEBRA

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5 = 10$

(a) Define a partial order relation. Give an example.

(b) Find the number of elements of order 5 in $\mathbb{Z}_{20}$.

(c) Find all cyclic sub-groups of the group $\{1, i, -1, -i\}$ with respect to multiplication.

(d) Prove or disprove "union of two sub-groups of a group $(G, \circ)$ is a sub-group of $(G, \circ)$".

(e) If G is a group and $a^2 = e$, $\forall a \neq e$. Prove that G is an abelian group.

(f) Is symmetric group S_3 cyclic? Give reasons.

(g) Examine whether $5\mathbb{Z} = \{5x : x \in \mathbb{Z}\}$ is an ideal or not of the ring $(\mathbb{Z}, +, \cdot)$, where $\mathbb{Z}$ is the set of all integers.

(h) Examine if the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & 2b \end{pmatrix} : a, b \in \mathbf{R}\right\}$ contains any divisor of zero.

2. (a) Let a relation R defined on the set $\mathbb{Z}$ by " $a\,R\,b$ if and only if $a-b$ is divisible by 5 for all $a, b \in \mathbb{Z}$. Show that R is an equivalence relation. 4

(b) Show that the set of all permutations on the set $\{1, 2, 3\}$ forms a non abelian group. 4

3. (a) Show that a non-empty subset H of G forms a subgroup of $(G, \circ)$ if and only if (i) $a \in H,\ b \in H \Rightarrow a \circ b \in H$, and $a \in H \Rightarrow a^{-1} \in H$. 4

(b) Prove that $\mathrm{SL}(n, \mathbb{R})$ is a normal subgroup of $\mathrm{GL}(n, \mathbb{R})$. 4

4. (a) Show that every proper sub-group of a group of order 6 is cyclic. 4

   (b) Prove that the commutator sub-group of any group is a normal sub-group. 4

5. (a) Prove that the order of every subgroup of a finite group G is a divisor of the order of G. 4

   (b) Prove that quotient group of an abelian group is abelian. Is the converse true? Justify. 4

6. (a) Prove that for any positive integer n, the set $U(n) = \{[x] : x$ is positive integer less than n and prime to $n\}$ is a group with respect to 'Multiplication Modulo n'. 4

   (b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ be defined by $f(x) = e^x$, $x \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Prove that f is invertible and find f^{-1}. 4

7. (a) Prove that any finite subgroup of the group of non zero complex numbers under multiplication is a cyclic group. 4

   (b) Let $G = S_3$ be a group and $H = \{\rho_0, \rho_1, \rho_2\}$ be a subgroup of G. Find all the left cosets of H (where the symbols have their usual meanings). 4

8. (a) Show that the ring of matrices $\left\{\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbf{Z}\right\}$ contains divisors of zeros and does not contain the unity. 4

   (b) Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}$ (the set of integers)$\}$. Show that $(\mathbb{Z}[\sqrt{2}], +, \cdot)$ is an integral domain. 4

9. (a) Prove that the ring of matrices $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbf{R}\right\}$ is a field. 4

   (b) Prove that a field is an integral domain. 4

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WEST BENGAL STATE UNIVERSITY
B.Sc. Honours 4th Semester Examination, 2023

MTMACOR08T-MATHEMATICS (CC8)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Let $f:[0,1]\to\mathbb{R}$ be defined by

$$f(0)=0,$$
$$f(x)=(-1)^n,\ \frac{1}{n+1}<x\leq\frac{1}{n},\ n=1,2,3,\ldots$$

Show that f is integrable on [0, 1].

(b) Let $f:[0,2]\to\mathbb{R}$ be a function defined by

$$f(x)=2x,\quad 0\leq x\leq 1$$
$$=x^2,\quad 1<x\leq 2$$

Show that f has no primitive although f is integrable on [0, 2].

(c) Find the values of p, if any, so that the integral

$$\int_1^\infty \frac{dx}{x^p} \text{ is convergent.}$$

(d) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{(n+1)x^n}{(n+2)(n+3)}.$$

(e) Test the uniform convergence of the sequence of functions $\{f_n\}$ on [0, 1] defined by $f_n(x)=x^n(1-x)$, $0\leq x\leq 1$.

(f) Verify whether the series $\sum_{n=1}^{\infty}\frac{x}{n(n+1)}$ converges uniformly in $[0,a]$ where $a>0$.

(g) Justify true or false: The function $f(x)=\sin x$, $0\leq x\leq\pi$, can be expressed as a Fourier cosine series.

(h) If the power series $\sum_{n=1}^{\infty}a_nx^n$ is convergent for all $x\in\mathbb{R}$ find the value of

$$\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}.$$

2. (a) (i) Prove that a monotone function f defined on a closed interval $[a, b]$ is integrable in the sense of Riemann. 2+2

(ii) Show that the function $f:[0, n] \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{[x]+1}, \quad 0 \le x \le n,$$

where $n \in \mathbb{N}$, $n > 1$, is R-integrable.

(b) If f be integrable on $[a, b]$ then show that the function F defined by 4

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b]$$

is continuous on $[a, b]$.

3. (a) Show that the integral 4

$\int_0^1 x^{m-1}(1-x)^{n-1}dx$ converges if and only if $m > 0$, $n > 0$.

(b) Show that the integral $\int_0^\infty \frac{x^{p-1}}{1+x}dx$ is convergent only when $0 < p < 1$. 4

4. (a) If for each $n \in \mathbb{N}$, $f_n:[a, b] \to \mathbb{R}$ be a function such that $f_n'(x)$ exists for all $x \in [a, b]$; $\{f_n(c)\}_n$ converges for some $c \in [a, b]$ and the sequence $\{f_n'\}_n$ converges uniformly in $[a, b]$, then prove that the sequence $\{f_n\}_n$ converges uniformly on $[a, b]$. 4

(b) The function f_n on $[-1, 1]$ are defined by $f_n(x) = \frac{x}{1+n^2x^2}$. Show that $\{f_n\}$ converges uniformly and that its limit function f is differentiable but the equality $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$ does not hold for all $x \in [-1, 1]$. 4

5. (a) Let g be a continuous function defined on [0, 1]. For each n in $\mathbb{N}$ define $f_n(x) = x^n g(x)$, $x \in [0, 1]$. Find a condition on g for which the sequence $\{f_n\}$ converges uniformly. 4

(b) If the series $\sum f_n$ converges uniformly in an interval $[a, b]$ prove that the sequence $\{f_n\}$ converges uniformly to the constant function 0 in $[a, b]$. 4

6. (a) Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$. 4

(b) Show that improper integral $\int_0^\infty \frac{\sin x}{x}dx$ is convergent. 4

7. (a) Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a given power series and $\mu = \overline{\lim} \, |a_n|^{1/n}$. Then show that the series is everywhere convergent if $\mu = 0$. 4

(b) Assuming $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ for $-1 < x < 1$, obtain the power series expansion for $\tan^{-1} x$. Also deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$. 3+1

8. Show that the function $f : [-\pi, \pi] \to \mathbb{R}$ be defined by 2+3+1+2

$$f(x) = \begin{cases} \cos x & 0 \le x \le \pi \\ -\cos x & -\pi \le x < 0 \end{cases}$$

satisfies Dirichlet's condition in $[-\pi, \pi]$. Obtain the Fourier co-efficients and the Fourier series for the function $f(x)$. Hence find the sum of the series

$$\frac{2}{1.3} - \frac{6}{5.7} + \frac{10}{9.11} - \cdots\cdots$$

9. (a) Let $f_n(x) = \frac{nx}{1+nx}$, $x \in [0, 1]$, $n \in \mathbb{N}$. Then show that 4

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx,$$

but $\{f_n\}_n$ is not uniformly convergent on [0, 1].

(b) Prove that the even function $f(x) = |x|$ on $[-\pi, \pi]$ has cosine series in Fourier's form as 3+1

$$\frac{\pi}{2} - \frac{4}{\pi}\left\{\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\cdots\right\}$$

Show that the series converges to $|x|$ in $[-\pi, \pi]$.

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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2023

MTMACOR09T-MATHEMATICS (CC9)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any *five* questions from the following: $2\times5 = 10$

(a) If S be the set of all points (x, y, z) in $\mathbb{R}^3$ satisfying the inequality $x+y+z<1$, determine whether or not S is open. 2

(b) Is the set $\mathbb{R}^n$ open? Justify your answer. 2

(c) Find the closure of $\{(x, y):1<x^2+y^2<2\}$. 2

(d) When a rational function $f(x)=\dfrac{P(x)}{Q(x)}$ (where P, Q are polynomials in the components of x) is continuous at each point x? 2

(e) Show that the function $f(x,y)=|x|+|y|$, $(x, y)\in\mathbb{R}^2$ possesses an extreme value at (0, 0) although $f_x(0,0)$, $f_y(0,0)$ do not exist. 2

(f) Find the gradient vector at each point at which it exists for the scalar field defined by $f(x, y)=x^2+y^2\sin(xy)$. 2

(g) Find $\iint\limits_R x^2dxdy$ where R is the region bounded by $x=0$, $y=0$ and $y=\cos x$. 2

(h) Use Green's theorem to compute the work done by the force field $f(x, y)=(y+3x)i+(2y-x)j$ in moving a particle once around the ellipse $4x^2+y^2=4$ in the counterclockwise. 2

2. (a) Show that the function is discontinuous at (0, 0) 4

$$f(x, y)=\begin{cases}\dfrac{x^3+y^3}{x-y}, & \text{when } x\neq y\\ 0\ , & x=y\end{cases}$$

(b) If $f(x, y)$ is continuous at (a, b) and $f(a, b)\neq 0$ then prove that there exists a neighbourhood of (a, b) where $f(x, y)$ and $f(a, b)$ maintain the same sign. 4

3. (a) The scalar field is defined by 1+1+1+1

$$f(x, y)=\begin{cases} 3y, & \text{when } x=y\\ 0\ , & \text{otherwise}\end{cases}$$

Do the partial derivatives $D_1f(0,0)$ and $D_2f(0,0)$ exist? If exist find their values. Find the directional derivative at the origin in the direction of the vector $i+j$.

(b) Evaluate $\iint\limits_R (x+2y)\,dxdy$, over the rectangle $R=[1, 2, 3, 5]$. 4

4. (a) Show that if $xyz = a^2(x+y+z)$, then the minimum value of $xy+zx+zy$ is $9a^2$. 4

(b) A function f is defined on the rectangle $R = [0,1;0,1]$ as follows: 4

$$f(x,y) = \begin{cases} \frac{1}{2}, & \text{when } y \text{ is rational} \\ x\ , & \text{when } y \text{ is irrational} \end{cases}$$

Show that the double integral $\iint_R f(x,y)\,dxdy$, does not exist.

5. (a) If $lx+my+nz=1$, l, m, n are positive constants, show that the stationary value of $xy+yz+zx$ is $(2lm+2mn+2nl-l^2-m^2-n^2)^{-1}$. 4

(b) For the vector field $F(x,y,z) = (x^2+yz)i + (y^2+xz)j + (z^2+xy)k$ compute the curl and divergence. 2+2

6. (a) Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{8abc}{3\sqrt{3}}$ (using Lagrange's method of multiplier). 4

(b) Let $y = F(x,t)$, where F is a differentiable function of two independent variables x and t which are related to two variables u and v by the relations $u = x+ct$, $v = x-ct$ $(c = \text{constant} \neq 0)$. Prove that the partial differential equation $\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$ can be transformed into $\frac{\partial^2 y}{\partial u \partial v} = 0$. 4

7. (a) Evaluate $\iint_E (x^2+y^2)\,dxdy$ over the region E bounded by $xy=1$, $y=0$, $y=x$, $x=2$. 4

(b) Show that $\iiint_E z^2\,dxdydz$, where E is the region of the hemisphere $z \geq 0$, $x^2+y^2+z^2 \leq a^2$, is $\frac{2}{15}\pi a^5$. 4

8. (a) Show that the entire volume bounded by the positive side of the three co-ordinate planes and the surface $\left(\frac{x}{a}\right)^{1/2} + \left(\frac{y}{b}\right)^{1/2} + \left(\frac{z}{c}\right)^{1/2} = 1$ is $\frac{abc}{90}$. 4

(b) Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ around the triangle OPQ whose vertices are $O(0,0,0)$, $P(2,0,0)$ and $Q(2,1,1)$, where $\vec{F} = (2x^2+y^2)\hat{i} + (3y-4z)\hat{j} + (x-y+z)\hat{k}$. 4

9. (a) Using Stokes' theorem, evaluate $\oint_C (xydx + xy^2dy)$, where C is the square in the xy-plane with vertices $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$. 4

(b) Using Green's theorem, evaluate $\int_\Gamma \{(y-\sin x)\,dx + \cos x\,dy\}$ where Γ is the triangle enclosed by the lines $y=0, x=\pi$ and $y=\frac{2x}{\pi}$. 4

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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2023

MTMACOR10T-MATHEMATICS (CC10)

RING THEORY AND LINEAR ALGEBRA-I

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any *five* questions from the following: $2\times 5 = 10$

(a) If in a ring R, $a^2 = a$ for all $a \in R$, prove that $a+b=0 \Rightarrow a=b$ for all $a, b \in R$.

(b) Let R be a ring with 1. Show that if R is a division ring, then R has no non-trivial ideal.

(c) Show that the characteristic of an integral domain D is either zero or a prime.

(d) Let f be a homomorphism of a ring R into a ring R'. Prove that $f(R)=\{f(a): a\in R\}$ is a subring of R'.

(e) Let $S=\{(x,y): x,y\in\mathbb{R}\}$. For $(x,y)\in S, (s,t)\in S$ and $c\in\mathbb{R}$, define $(x,y)+(s,t) = (x+s, y-t)$ and $c(x,y)=(cx,cy)$. Is S a vector space over $\mathbb{R}$? — Justify.

(f) Let V be a vector space of real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$W=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in V: a+b=0\right\}$. Prove that W is a subspace of V.

(g) Find the dimension of the subspace S of the vector space $\mathbb{R}^3$ given by $S=\{(x,y,z)\in\mathbb{R}^3: 2x+y-z=0\}$.

(h) Define $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by $T(f(x))=f'(x)$, where $f'(x)$ denotes the derivative of $f(x)$. Show that T is a linear transformation.

2. (a) Find all subrings of the ring $\mathbb{Z}$ of integers. 4

(b) Let R be a commutative ring with 1 and M be an ideal of R. Show that M is a maximal ideal if and only if R/M is a field. 4

3. (a) Show that $\mathbb{Z}[\sqrt{3}]=\{a+b\sqrt{3}: a,b\in\mathbb{Z}\}$ is an integral domain but not a field. 2+2

(b) Let $n\in\mathbb{Z}$ be a fixed positive integer. If $\mathbb{Z}/\langle n\rangle$ is a field, then show that n is prime, where $\langle n\rangle=\{qn: q\in\mathbb{Z}\}$ and $\mathbb{Z}/\langle n\rangle=\{a+\langle n\rangle: a\in\mathbb{Z}\}$. 4

4. (a) Prove that the cancellation law holds in a ring $(R, +, \cdot)$ if and only if $(R, +, \cdot)$ contains no divisor of zero. 4

(b) If $(R, +, \cdot)$ is an integral domain of prime characteristic p then prove that $(a+b)^p = a^p + b^p$, for all $a, b \in R$. 4

5. (a) Let A be an ideal of a ring R. Define $f : R \to R/A$ by $f(r) = r + A$, for all $r \in R$. Prove that f is a ring homomorphism. 3

(b) If f is a homomorphism of a ring R into a ring S then prove that $R/\ker f \simeq f(R)$. 5

6. (a) Let W_1, W_2 be two subspaces of a vector space V over a field $\mathbb{F}$. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. 4

(b) Let $W = \{(x, y, z) \in \mathbb{R}^3 : x - 4y + 3z = 0\}$. Show that W is a subspace of $\mathbb{R}^3$. Also find a basis of W. 2+2

7. (a) Let V be a vector space over a field $\mathbb{F}$, with a basis consisting of n elements. Then show that any $n+1$ elements of V are linearly dependent. 4

(b) Let V be a vector space of dimension m and W be a vector space of dimension n over a field $\mathbb{F}$. 4

Prove that $\dim(V/W) = m - n$.

8. (a) Let V and W be the vector spaces over the field F and let $T : V \to W$ be a linear transformation. If V is of finite dimension then prove that 5

$$\dim(V) = \dim(\ker T) + \dim(\operatorname{Im} T)$$

(b) Find the linear transformation $T : R^2 \to R^2$ such that $T(2, 3) = (2, 3)$ and $T(1, 0) = (0, 0)$. 3

9. (a) Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by 4

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases for $P_2(\mathbb{R})$ and $\mathbb{R}^3$ respectively. Compute $[U]_\beta^\gamma$, $[T]_\beta$ and $[UT]_\beta^\gamma$.

(b) Determine whether the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by 4

$$T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$$

is invertible and justify your answer.

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